Optimization

Homework 1

(Due Day: 9:00 AM, Oct 22, 2008, hardcopies in the class)

1. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given below:

$$f(\boldsymbol{x}) = \boldsymbol{x}^T \begin{bmatrix} 1 & 2 \\ 4 & 7 \end{bmatrix} \boldsymbol{x} + \boldsymbol{x}^T \begin{bmatrix} 3 \\ 5 \end{bmatrix} + 6.$$

a. Find the gradient and Hessian of f at the point $[1, 1]^T$.

b. Find the directional derivative of f at $[1, 1]^T$ with respect to a unit vector in the direction of maximal rate of increase.

c. Find a point that satisfies the FONC (interior case) for f. Does this point satisfy the SONC (for a minimizer)?

2. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given below:

$$f(\boldsymbol{x}) = \boldsymbol{x}^T \begin{bmatrix} 2 & 5 \\ -1 & 1 \end{bmatrix} \boldsymbol{x} + \boldsymbol{x}^T \begin{bmatrix} 3 \\ 4 \end{bmatrix} + 7.$$

a. Find the directional derivative of f at $[0, 1]^T$ in the direction $[1, 0]^T$.

b. Find all points that satisfy the first-order necessary condition for f.

Does f have a minimizer? If it does, then find all minimizer(s); otherwise explain why it does not.

3. Consider the problem

$$\begin{array}{ll} \text{minimize} & f(\boldsymbol{x}) \\ \text{subject to} & \boldsymbol{x} \in \Omega, \end{array}$$

where $f : \mathbb{R}^2 \to \mathbb{R}$ is given by $f(\boldsymbol{x}) = 5x_2$ with $\boldsymbol{x} = [x_1, x_2]^T$, and $\Omega = \{\boldsymbol{x} = [x_1, x_2]^T : x_1^2 + x_2 \geq 1\}$. Answer each of the following questions, showing complete justification.

a. Does the point $\boldsymbol{x}^* = [0, 1]^T$ satisfy the first-order necessary condition?

b. Does the point $\boldsymbol{x}^* = [0, 1]^T$ satisfy the second-order necessary condition?

c. Is the point $\boldsymbol{x}^* = [0, 1]^T$ a local minimizer?

4. Consider the problem

$$\begin{array}{ll} \text{minimize} & f(\boldsymbol{x}) \\ \text{subject to} & \boldsymbol{x} \in \Omega, \end{array}$$

where $\boldsymbol{x} = [x_1, x_2]^T$, $f : \mathbb{R}^2 \to \mathbb{R}$ is given by $f(\boldsymbol{x}) = 4x_1^2 - x_2^2$, and $\Omega = \{\boldsymbol{x} : x_1^2 + 2x_1 - x_2 \geq 0, x_1 \geq 0, x_2 \geq 0\}$.

a. Does the point $\boldsymbol{x}^* = \mathbf{0} = [0, 0]^T$ satisfy the first-order necessary condition?

b. Does the point $\boldsymbol{x}^* = \mathbf{0}$ satisfy the second-order necessary condition?

c. Is the point $\boldsymbol{x}^* = \mathbf{0}$ a local minimizer of the given problem?

5.

Let $f(x) = x^2 + 4\cos x$, $x \in \mathbb{R}$. We wish to find the minimizer x^* of f over the interval $[1, 2]$. (*Calculator users:* Note that in $\cos x$, the argument x is in radians). Apply Newton's method, using the same number of iterations as in part b, with $x^{(0)} = 1$.

Iteration k	a_k	b_k	$f(a_k)$	$f(b_k)$	New uncertainty interval
1	?	?	?	?	[?,?]
2	?	?	?	?	[?,?]
⋮	⋮	⋮	⋮	⋮	⋮

(part b)

6.

The objective of this exercise is to implement the secant method
Let $g(x) = (2x - 1)^2 + 4(4 - 1024x)^4$. Find the root of $g(x) = 0$ using the secant method with $x^{(-1)} = 0$, $x^{(0)} = 1$, and $\varepsilon = 10^{-5}$. Also determine the value of g at the obtained solution.